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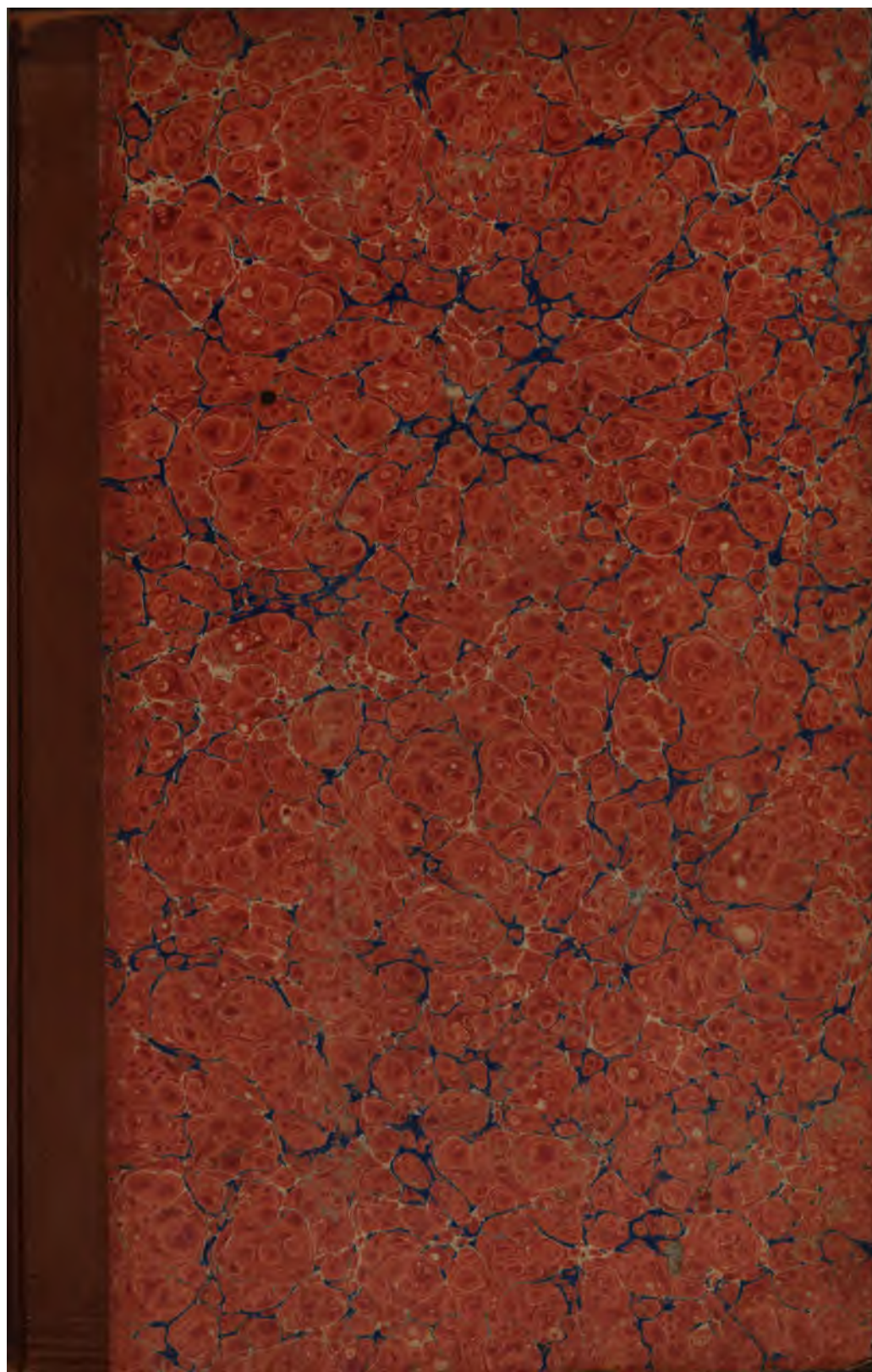
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CONIC SECTIONS:

THEIR PRINCIPAL PROPERTIES PROVED
GEOMETRICALLY.



BY

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P R E F A C E.

I SHOULD not have published the following pages, if it had not been represented to me that there is not at present any geometrical Treatise on Conic Sections, which the Students of the University can conveniently use. Mr. Hustler, in his last edition, has *defined* the Conic Sections by their being Sections of a Cone, cut by a plane in certain manners; and thus he makes it necessary for his readers to begin the subject with Solid Geometry. This is an inconvenience which, I think, we in this University have always endeavoured to avoid, although the method has the sanction of history and antiquity in its favour. The subject is rendered more simple, and better suited to its place as an immediate sequel to Plane Geometry, when we define the figures by their plane properties, as I have done.

The study of Conic Sections in this form appears to be an indispensable part of a course of mathematical reading which goes beyond Elementary Geometry; and especially of a course which includes any study of Newton's *Principia*. The study of Conic Sections as a branch of Analytical Geometry can by no means supply the place of an acquaintance with the geometrical proofs. By seeing the subject presented only in an analytical form, the student obtains a very erroneous notion of the relation of the subject to geometry, and remains utterly ignorant of that part of mathematics which all mathematicians up to the present time have understood by the term *Conic Sections*. Nor can he acquire a proper apprehension of the peculiar beauty and generality of the language of Analytical Geometry, without seeing the properties of the Conic Sections *before* they are translated into that language. Moreover it is not too much to say that those students who have never become acquainted with the reasonings of Conic Sections in a geometrical form, will unavoidably have a very confused and imperfect comprehension of the reasoning of the *Principia* of Newton; and cannot help

looking upon the study of that work as a troublesome and incongruous excrescence in their mathematical course.

I am fully aware that those students of mathematics who proceed to the higher parts of the science must give a great portion of time to analytical studies: and that, in the common course of mathematical reading, no large space can be assigned to the subject here treated of. On that account, I have made the Treatise very brief, and have inserted none but what appear to be *classical* Propositions in Conic Sections. Nor indeed have I introduced all such Propositions. The mathematical reader who is acquainted with the subject, will here miss many properties of great beauty, which he will have seen elsewhere; and which I have sacrificed to the brevity which I thought desirable. For the same reason, when some properties of the Ellipse have been proved, I have omitted the analogous properties of the Hyperbola (as Props. xvi, xvii. of the Ellipse). The reader will have no difficulty in supplying the proof from the analogies of the other properties. For like reasons, the steps of Elementary Geometry which occur in the proofs are given in a very concise manner. The geometrical student will be able to unfold the reasoning more fully; and it will be an instructive exercise to do so.

In reference to the Propositions and their proofs, I may observe that I have thought it desirable to prove, for each curve, first, and independently, those properties which do not depend upon a *Tangent* to the curve; as the properties with reference to the Principal Axis. I have not attempted to give analogous proofs of the analogous Propositions of the Ellipse and Hyperbola with regard to their Diameters; (Ellipse, Prop. xii., and Hyperbola, Prop. xiv.) for the proofs are simplified in the two cases by relations special to each; the relations of the Ellipse to the Circumscribing Circle, and of the Hyperbola to its Asymptotes. In the other Propositions, the analogy of the demonstrations is preserved.

TRINITY LODGE,
May 13, 1846.

CONIC SECTIONS:

THEIR PRINCIPAL PROPERTIES PROVED GEOMETRICALLY.

INTRODUCTION.

(1). By calling proofs *geometrical*, we mean that they are deduced from the propositions in Euclid's Elements of Geometry, by reasoning of the same kind as that which is used in the Elements.

But though this is the method pursued in the following pages, I have employed some modes of expression which are not used in the common editions of Euclid; and have also assumed a few simple geometrical propositions which do not stand in the Elements. I will enumerate the principal points of this kind.

(2). I have used the algebraical signs $=$, $+$ and $-$. These are to be understood as being only marks for words, not symbols by means of which operations are performed.

Thus $A + B$ denotes " A together with B ;" $A - B$ may be read " A , its excess over B ." And in order to preserve the geometrical character of the reasoning, we must not apply indiscriminately the algebraical rule of transposing terms and changing their signs. Thus, if $A - B = C - D$, we must not infer at once that $A + D = C + B$. But we may deduce this result by two steps, thus: $A - B + D = C$, by adding D to the equals: and hence, $A + D = C + B$, by adding B to the equals. This is not a merely fanciful distinction; for most minds moderately familiar with geometrical reasoning, can follow one of these steps at a time, but not both together.

If $A - B = 2P$, and $A + B = 2Q$; then $A = P + Q$, $B = Q - P$.

(3). Also $A \cdot B$ or AB is used to express the rectangle of lines A and B , and A^2 to express the square of A . And the propositions of Euclid's Second and other Books are expressed by means of this notation. For instance, Eucl. II. 3. is thus expressed; $A(B + A) = AB + A^2$: and Eucl. II. 4. thus; $(A + B)^2 = A^2 + B^2 + 2AB$. Also Eucl. II. 5. Cor. is thus expressed; $A^2 - B^2 = (A + B)(A - B)$. And to express Eucl. II. 7; if A be the whole line, B one of the parts, $A - B$ the other part; we have $A^2 + B^2 = 2AB + (A - B)^2$; and hence taking away $2AB$ from both,

$$A^2 + B^2 - 2AB = (A - B)^2,$$

which proposition we shall also assume.

(4). Proportion is expressed in the usual manner,

$$A : B :: C : D.$$

And the propositions of Euclid's Fifth Book (including Simson's) are represented accordingly. Thus

$$\text{if } A : B :: C : D,$$

$$\text{Eucl. v. 9, } B : A :: D : C; (\textit{invertendo}).$$

$$\text{Also Eucl. v. 16, } A : C :: B : D; (\textit{alternando}).$$

$$\text{Also Eucl. v. 17, } A : A - B :: C : C - D; (\textit{dividendo}).$$

$$\text{Also Eucl. v. 18, } A + B : B :: C + D : D (\textit{componendo}).$$

$$\text{Also Eucl. VI. 16, if } AD = BC, A : B :: C : D.$$

Other consequences of the proposition $A : B :: C : D$ are also assumed, and might easily be proved; namely,

$$A + B : A - B :: C + D : C - D;$$

$$A - B : B :: C - D : D;$$

$$A^2 : B^2 :: C^2 : D^2;$$

$$AM : BM :: CN : DN; \text{ and the like.}$$

(5). The composition of proportions by algebraical multiplication may be justified by geometrical reasoning; and is here assumed. Thus

if $A : B :: C : D$,
 and $M : N :: D : E$,
 and $N : A :: P : C$;

we infer that $M : B :: P : E$ (Ellipse, Prop. x1.);
 which we may thus prove. The composition of the two first
 proportions gives us, by Eucl. vi. 14, and v. 22,

$$AM : BN :: C : E,$$

and the third proportion gives us

$$MN : AM :: P : C;$$

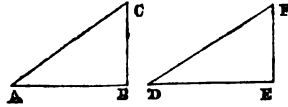
whence by Eucl. v. 22,

$$MN : BN :: P : E;$$

that is, $M : B :: P : E$.

The following Propositions may be assumed.

(6). If ABC , DEF be two triangles, right-angled at B
 and E , and having the sides AB ,
 AC equal respectively to DE ,
 DF ; the triangles are equal in all
 respects.



For $BC^2 = AC^2 - AB^2$, and $EF^2 = DF^2 - DE^2$; when
 $BC^2 = EF^2$; and $BC = EF$; and hence, by Eucl. i. 8, the
 triangles are equal in all respects.

(7). In any triangle, if the base be divided into segments
 by a perpendicular from the opposite angle;

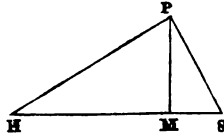
base : sum of sides :: difference of sides : difference of segments.

Let HPS be a triangle, PM the perpendicular falling
 upon HS in M ; then $HM^2 + MP^2 = HP^2$; and $SM^2 + MP^2$
 $= SP^2$, by Eucl. i. 47.

Hence $HM^2 - SM^2 = HP^2 - SP^2$.

Hence, $(HM + SM)(HM - SM)$
 $= (HP + SP)(HP - SP)$,

by Eucl. ii. 5.



Hence Eucl. vi. 16,

$$HM + SM : HP + SP :: HP - SP : HM - SM,$$

and $HM + SM$ is the base HS . Therefore, &c.

(8). If the perpendicular fall without the base, we have

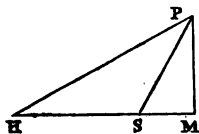
base : sum of sides :: diff. of sides : sum of segments.

For, as before, $HM^2 + MP^2 = HP^2$; $SM^2 + MP^2 = SP^2$;

whence $HM^2 - SM^2 = HP^2 - SP^2$;

whence $(HM + SM)(HM - SM)$

$$= (HP + SP)(HP - SP).$$

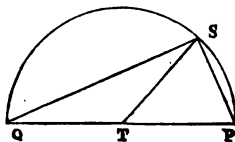


Hence, $HM - SM : HP + SP :: HP - SP : HM + SM$;

and $HM - SM$ is the base HS . Therefore, &c.

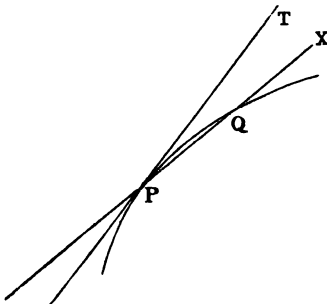
(9). If QSP be a triangle right-angled at S , and if QP be bisected in T , $TS = TQ$ or $= TP$.

For if a circle be described with center T and radius TQ or TP , it will pass through S , because QSP is the angle in a semicircle.



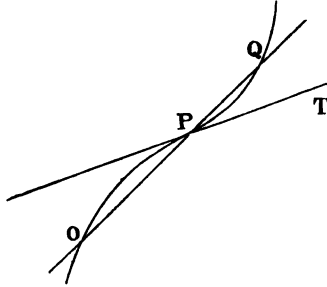
(10). If a straight line PQX meet a curve in two points P , Q , and if Q come up to P and *ultimately* coincide with it; the straight line PQX will *ultimately* assume a position PT ; in this position it is a *tangent* to the curve at the point P .

If the line PQ meet the curve in two points *only*, when Q comes up to P , Q and P become one point, and the line meets the curve in one point only.



(11). The tangent to a *circle* is defined to be “a straight line which meets the curve and being produced does not cut it.” (Eucl. III. Def. 2). But we cannot give this as a

definition of a tangent to *any* curve. For if the straight line which cuts the curve, cut it in three points, as O , P , Q ; and if, when Q comes to P , O also comes to P ; the straight line, though it then becomes a tangent, will cut the curve. This takes place at a point of *inflection* in a curve.



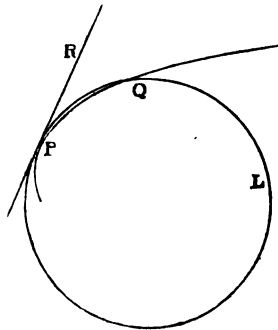
Hence, we take a definition of a tangent depending upon the *ultimate* coincidence of two points; or upon a *limit*. We may express the definition by saying that a *Tangent is the Limit of a Secant*.

(12). The notion of a Limit and its results are given in the First Section of Newton's *Principia*; and the proper place for the doctrine of Conic Sections in the order of reasoning is between the First and Second Sections of the *Principia*. But the only proposition of the First Section which we require, in order to prove the properties of tangents is this: that if, in two triangles ABC , DEF , (page 3), AB , AC be equal respectively to DE , DF , and B and E ultimately right angles; the triangles ACB , DEF will ultimately be equal in all respects.

(13). Two curves touch each other at any point, when they have the same tangent at that point.

(14). If a circle be drawn, touching a curve at any point, and cutting it in a second point; and if the second point approach indefinitely to the first, and coincide with it; the *Ultimate* or *Limiting Circle* thus obtained is the *Circle of Curvature* of the curve at that point.

If PQL be a circle which meets a curve in P , Q , the circle and curve having the same tangent PR ; and if Q come up to P , the circle becomes the circle of curvature of the curve at P .



CONIC SECTIONS.

The Conic Sections which we here consider are the *Parabola*, the *Ellipse*, and the *Hyperbola*.

THE PARABOLA.

DEF. 1. The Parabola is a curve such that every point in the curve is at the same distance from a given point, (called the *Focus*,) and a given straight line, (called the *Directrix*).

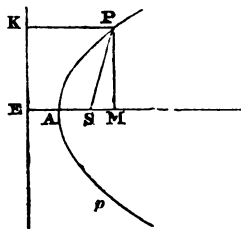
If S be the given point, EK the given straight line, P a point in the curve, PKE a right angle; we must have $SP = PK$, wherever be the point P .

PROP. I.

If SE be perpendicular to the directrix, and A bisect SE , the curve passes through the point A .

For $SA = AE$; which determines the point A to be in the curve, by Def. 1.

DEF. 2. The line AS produced is the *Principal Axis* of the parabola, and A is the *Vertex*.



PROP. II.

If from any point P in the parabola, PM be drawn perpendicular to the principal axis, AM , S being the focus; then (last fig.)

$$MP^2 = 4SA \cdot AM.$$

For* $EM^2 = AM^2 + AE^2 + 2EA \cdot AM$,
by Eucl. II. 4. ; also, since $SM = AM - AS$,

$$SM^2 = AM^2 + AS^2 - 2SA \cdot AM,$$

by Eucl. II. 7. See Intro. (3.)

* Proof otherwise. By Eucl. II. 8, $4SA \cdot AM + MS^2 = EM^2$, because $SA = AE$.
But $EM = PK = PS$, by Def. 1. Therefore $4SA \cdot AM + MS^2 = PS^2 = MP^2 + MS^2$.
Therefore $4SA \cdot AM = MP^2$.

Hence (since $AS = AE$) $EM^2 - SM^2 = 4SA \cdot AM$, by Introd. (2.) But $EM^2 - SM^2 = KP^2 - SM^2 = SP^2 - SM^2 = MP^2$; hence $MP^2 = 4SA \cdot AM$.

If M fall between S and A , $SM = SA - AM$, and the proof is in other respects the same.

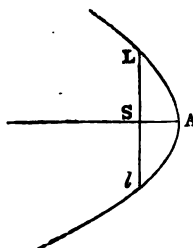
COR. If PM produced meet the curve again in p , we may in the same manner prove that $Mp^2 = 4AS \cdot AM$. Hence $Mp = MP$.

DEF. 3. The portion of the axis AM is called the *Abcissa* of the point P ; MP or Mp is the *Ordinate*: Pp is the *Double Ordinate*.

PROP. III.

The double ordinate drawn through the focus is $= 4AS$.

Let Ll be the double ordinate drawn through the focus. By Prop. II., SA being now the abscissa, $SL^2 = 4SA \cdot SA = 4SA^2$. Hence, $SL = 2SA$. For the same reason $Sl = 2SA$. Hence $Ll = 4SA$.

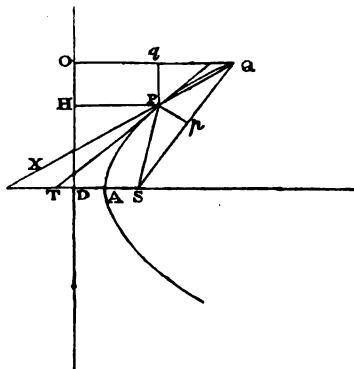


DEF. 4. The line $4SA$ or Ll is called the *Principal Parameter* or the *Latus Rectum*.

PROP. IV.

In the parabola, the tangent at any point bisects the angle made by the two straight lines, one drawn to the focus, and the other perpendicular to the directrix.

Let P be any point in the parabola, PS , PH the lines drawn to the focus and perpendicular to the directrix: Q a point near to P .



Take in SQ , $Sp = SP$; draw QO perpendicular to the directrix; Draw Pq perpendicular on QO .

By the Def. of the Parabola, $SP = PH$, and $SQ = QO$.

Hence $SQ - SP = QO - PH = QO - qO$; or $Qp = Qq$.

Also since $Sp = SP$, the angles SPp , SpP are always equal; and when Q comes up to P , and PSQ vanishes, each is ultimately equal to a right angle: therefore PpQ is ultimately a right-angled triangle. Also PqQ is a right-angled triangle.

Hence ultimately, when Q comes up to P , by Introd. Art. (12), the angles PQp , PQq are equal: and hence, if QP be produced to X ; the angles XPH , XPS are ultimately equal.

But ultimately, when Q comes up to P , the secant QPX becomes a tangent QT , by Int. Art. (10).

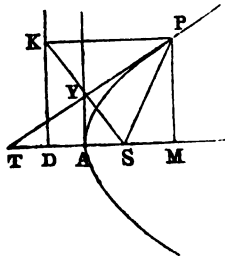
Hence $TPS = TPH$.

COR. If HP be produced to I , and TP to R , the angles IPR , SPT are equal. (See fig. p. 11.)

PROP. V.

In the parabola, if PT a tangent, PM an ordinate, meet the principal axis AS in T , M ; $AT = AM$.

Draw PK perpendicular to the directrix, and therefore parallel to the principal axis. Then since (Prop. IV.) $TPS = TPK$, and $TPK = PTS$; $TPS = PTS$; whence $ST = SP$, that is, $= PK$. But $PK = DM$. Hence $ST = DM$; and taking away DS , $DT = SM$; whence, adding $AD = AS$, $AT = AM$.



COR. 1. Hence $TM = 2AM$, and $TM = 2AT$.

COR. 2. If SY be perpendicular on PT , $PY = YT$, because $SP = ST$.

COR. 3. Hence $TA : AM :: TY : YP$; and AY is parallel to PM , (Eucl. VI. 2.) and therefore perpendicular to AM .

COR. 4. Hence the angles ASY , YSP are equal, and SAY , SYP are right angles; hence the triangles are similar; and

$$SA : SY :: SY : SP.$$

COR. 5. Hence $SY^2 = SA \cdot SP$.

COR. 6. If KY be joined, since $SP = PK$, the angle $SPY = KPY$, and PY is common; $KYP = SYP =$ a right angle by construction. Hence SYK is a straight line, and $SY = YK$.

DEF. 4. Any line parallel to the principal axis (Def. 2), is a *Diameter*, and the point where it meets the curve is the *Vertex* of that diameter.

DEF. 5. A line parallel to a tangent at the vertex and intersecting the diameter, is an *Ordinate* to the diameter: and the part cut off from the diameter is the *Abscissa*. Thus in Prop. VII, PR is a diameter, P , the vertex; QV , parallel to the tangent PY , is an ordinate; PV , the abscissa.

DEF. 6. A *Normal* is a line drawn at any point perpendicular to the tangent at that point.

PROP. VI.

In the parabola, if PK be a normal, meeting the principal axis in K , PM an ordinate; $MK = 2AS$.

For since PK is perpendicular to TP , by sim. tri.

$$TM : MP :: MP : MK;$$

or (Prop. v. Cor. 1.)

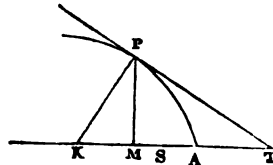
$$2AM : MP :: MP : MK.$$

$$\text{But } MP^2 = 4AS \cdot AM = 2AS \cdot 2AM;$$

by Prop. 11; hence by Int. (4),

$$2AM : MP :: MP : 2AS;$$

wherefore $MK = 2AS$.



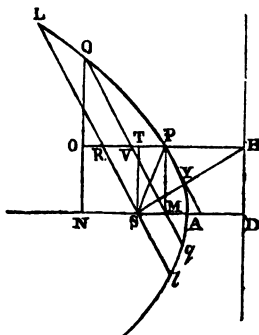
PROP. VII.

In the parabola, PV being the abscissa in any diameter and QV the ordinate,

$$4SP \cdot PV = QV^2.$$

Let VP meet the directrix in H ; join SH ; SH is perpendicular to the tangent PY , and $SY = YH$ (Prop. v. Cor. 6.) Draw SR parallel to PY , meeting PV in R . By similar triangles HPY , HRS , since $SY = YH$, $RP = PH$. But $SP = PH$, by Def. of Parabola. Hence $HR = 2HP = 2SP$.

Draw ST perpendicular to HR and PM to AS ; and let SA meet the directrix in D . And let QN , perpendicular to the axis in N , meet PV in O .



$$\text{Then } NQ^2 = NO^2 + OQ^2 + 2NO \cdot OQ.$$

But $NQ^2 = 4AS \cdot AN$, and NO^2 , that is, $MP^2 = 4AS \cdot AM$. (Prop. II.)

$$\text{Hence } 4AS \cdot AN = 4AS \cdot AM + OQ^2 + 2NO \cdot OQ.$$

$$\text{Or, taking away equals, } 4AS \cdot MN = OQ^2 + 2NO \cdot OQ.$$

But by similar triangles QOV , HTS , (both similar to STR)

$$OQ : OV :: TH : TS, \text{ that is, } OQ : OV :: 2AS : NO.$$

Therefore $NO \cdot OQ = 2AS \cdot OV$, and $2NO \cdot OQ = 4AS \cdot OV$.

Hence $4AS \cdot MN$, that is, $4AS \cdot OP = OQ^2 + 4AS \cdot OV$; and taking away equals, $4AS \cdot PV = OQ^2$.

But by similar triangles, $OQ^2 : QV^2 :: HS^2 : HR^2$, that is, $HT \cdot HR : HR^2$, or $HT : HR$, (Eucl. VI. 8.)

that is $2AS : 2SP$, as shewn before;

$$\therefore OQ^2 : QV^2 :: 4AS \cdot PV : 4SP \cdot PV.$$

But $4AS \cdot PV = OQ^2$, as shewn already;

$$\therefore (\text{Eucl. v. 14.}) 4SP \cdot PV = QV^2.$$

COR. In the same way, if the ordinate QV meet the curve again in q , we may prove $4SP \cdot PV = qV^2$.

Hence $qV = QV$.

DEF. 7. A line $4SP$ is called the *Parameter* to the vertex P , or to the diameter PR .

PROP. VIII.

For any diameter, the ordinate passing through the focus is equal to the parameter to that diameter.

If PY (last fig.) be the tangent, $SPY = HPY$ by Prop. iv.; and Ll is parallel to PY ; therefore by alt. angles $PSR = PRS$; and $PR = PS$. Now $RL^2 = 4SP \cdot PR$ by Prop. vii.;

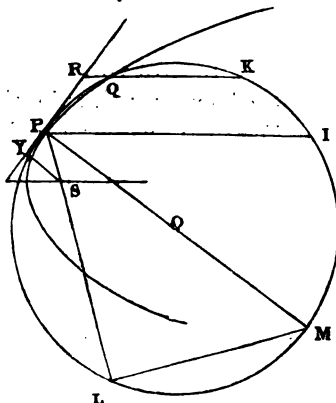
$$\therefore RL^2 = 4SP^2; RL = 2SP, Ll = 4SP.$$

DEF. 8. The radius of the circle of curvature (Int. (14.)) is called *the Radius of Curvature*. The chord of the circle of curvature drawn parallel to the axis, or passing through the focus, is called *the Chord of Curvature* parallel to the axis or through the focus.

PROP. IX.

In the parabola, at any point P , the chord of curvature parallel to the axis $= 4SP$.

Let PQK be a circle which has the tangent PR in common with the parabola at P ; and which cuts the parabola in Q , near to P . Draw RQK parallel to the axis, cutting the circle in K .



If QV were drawn as in Prop. vii., parallel to PR , meeting the diameter PI in V , it is evident that $QV = RP$ and $PV = RQ$. And since by that Prop. $4SP \cdot PV = QV^2$, we have $4SP \cdot RQ = RP^2$. But by

Eucl. III. 36. $RK \cdot RQ = RP^2$; and therefore $RK = 4SP$. And since this is always true, it is true when Q comes up to P . Therefore ultimately $RK = 4SP$: that is, for the ultimate circle PI , parallel to AS , is $= 4SP$.

But the ultimate circle is the circle of curvature at P . (Int. Art. 14.) Hence PI , the chord of the circle of curvature parallel to the axis, is $= 4SP$.

COR. 1. If PL be the chord of this circle drawn through the focus, PL and PI make equal angles with the tangent. (Prop. IV., Cor.) Hence if IL were joined, angle $PLI = PIL$ (Eucl. III. 32.) Hence $PL = PI$. Wherefore PL , the chord of curvature drawn through the focus, $= 4SP$.

COR. 2. Let PM be drawn perpendicular to the tangent at P . Then PM will be a diameter of the circle of curvature. And if LM be joined, the triangles PLM , PYS will evidently be similar. Hence

$SY : SP :: PL : PM$; that is, $SY : SP :: 4SP : PM$; PM being the diameter of the circle of curvature.

COR. 3. If O bisect PM , $SY : SP :: 2SP : PO$; the radius of curvature.

THE ELLIPSE.

DEF. 1. An ellipse is a curve line such that the distances of every point in the curve from two fixed points (the *Foci*) are together equal to a constant line.

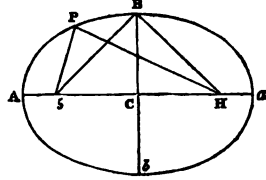
If S, H , be the two fixed points; P , a point in the curve; then $SP + PH =$ a constant line, wherever be P .

PROP. I.

If SH be joined and bisected in C , and if A , in HS produced, be a point in the curve, $SP + PH = 2CA$.

For since A, P are points in the curve, $SP + PH = SA + AH$.
 But $SA + AH = CA - CS + CA + CH$;
 and this is $= 2CA$, because $CS = CH$.

COR. 1. If a , in SH produced, be a point in the curve, it will appear, in like manner, that $SP + PH = 2Ca$.
 Hence $Ca = CA$.



COR. 2. Hence $SP + PH = Aa$.

DEF. 2. C is called the *Center* of the ellipse. Aa is called the *Axis Major*; CA is called the *Semi-axis Major*. The ratio of CS to CA is called the *Excentricity*.

COR. 3. If CB be perpendicular to CA , and B a point in the curve, $CB^2 = CA^2 - CS^2$.

For, since B is a point in the curve, by Def. 1, $SB + BH = 2CA$.

And $BH = SB$, by equal triangles.

Therefore, $2SB = 2CA$, and $SB = CA$.

And $CB^2 = SB^2 - CS^2$; $\therefore CB^2 = CA^2 - CS^2$.

The line CB is called the *Semi-axis Minor*.

COR. 4. If b in BC produced be a point in the curve, in like manner $Cb^2 = CA^2 - CS^2$; whence $Cb = CB$, and $Bb = 2CB$.

The line Bb is the *Axis Minor*.

COR. 5. Hence $CA^2 = CB^2 + CS^2$.

COR. 6. $CB^2 = AS \cdot Sa$, by Cor. 3; for $CA^2 - CS^2 = AS \cdot Sa$. (Int. (3.))

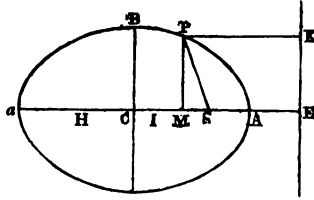
PROP. II.

In the ellipse, if from the point P , we draw PM perpendicular on the axis major, and take a point I , such that

$$CA : CS : CM : CI;$$

we have $AI = SP$, and $aI = HP$.

For $SP^2 = SM^2 + MP^2$, and $HP^2 = HM^2 + MP^2$;



therefore $HP^2 - SP^2 = HM^2 - SM^2$;

therefore $(HP + SP)(HP - SP) = (HM + SM)(HM - SM)$;

that is, $2CA \cdot (HP - SP) = 2CS \cdot 2CM$.

Whence $CA : CS :: 2CM : HP - SP$. Int. (4.)

But $CA : CS :: 2CM : 2CI$, by construction;

whence by Eucl. v. 14. $HP - SP = 2CI$.

But $HP + SP = 2CA$ by Prop. 1.

Hence by Int. (3.) $HP = CA + CI = aC + CI = aI$;

and $SP = CA - CI = AI$.

PROP. III.

If in the axis major of an ellipse, we take $CS : CA :: CA : CE$, and draw EK perpendicular to EC , meeting PK perpendicular to EK from a point P in the curve; $SP : PK$ is a constant ratio. (See last fig.)

We have $CE : CA :: CA : CS$, that is $:: CM : CI$, retaining the construction of Prop. II.

Hence $CE : CM :: CA : CI$,

whence $CE : CE - CM :: CA : CA - CI$; (Int. 4.)

or $CE : ME :: CA : AI$;

that is, $CE : PK :: CA : SP$, by Prop. II.

Hence $SP : PK :: (CA : CE ::) CS : CA$, a constant ratio.

DEF. 4. EK is called the *Directrix* of the ellipse.

PROP. IV.

In the ellipse, MP being an ordinate, (fig. Prop. II.)

$$AM \cdot Ma : MP^2 :: CA^2 : CB^2.$$

Take $CA : CS :: CM : CI$; whence $CA \cdot CI = CS \cdot CM$.

Now by Prop. II. $SP = AI$, whence $SP^2 = AI^2$.

$$\text{But } SP^2 = SM^2 + MP^2; \therefore AI^2 = SM^2 + MP^2;$$

$$\text{that is, } (CA - CI)^2 = (CS - CM)^2 + MP^2;$$

$$\text{hence, } CA^2 + CI^2 - 2CA \cdot CI = CS^2 + CM^2 - 2CS \cdot CM + MP^2.$$

Hence, (adding equals $2CA \cdot CI$ and $2CS \cdot CM$)

$$CA^2 + CI^2 = CS^2 + CM^2 + MP^2.$$

But $CA^2 = CB^2 + CS^2$; by Prop. I. Cor. 5.;

$$\therefore CB^2 + CS^2 + CI^2 = CS^2 + CM^2 + MP^2;$$

$$\therefore CB^2 + CI^2 = CM^2 + MP^2;$$

$$\therefore CB^2 - MP^2 + CI^2 = CM^2;$$

$$\therefore CB^2 - MP^2 = CM^2 - CI^2.$$

But since $CA : CS :: CM : CI$, $CA^2 : CS^2 :: CM^2 : CI^2$; (Int. 4).

$$\text{whence } CA^2 : CA^2 - CS^2 :: CM^2 : CM^2 - CI^2;$$

$$\text{therefore } CA^2 : CA^2 - CS^2 :: CM^2 : CB^2 - MP^2;$$

that is, $CA^2 : CB^2 :: CM^2 : CB^2 - MP^2$ (Prop. I. Cor. 3).

$$\text{whence } CA^2 : CM^2 :: CB^2 : CB^2 - MP^2;$$

$$\text{whence } CA^2 : CA^2 - CM^2 :: CB^2 : MP^2;$$

$$\text{or } CA^2 : AM \cdot Ma :: CB^2 : MP^2; \text{ (Int. (3.))}$$

$$\text{whence } AM \cdot Ma : MP^2 :: CA^2 : CB^2.$$

If M fall between S and A , $SM = CM - CS$, and the proof is in other respects the same.

COR. 1. If MP' be the ordinate on the other side of the axis major, we may prove in like manner that

$$AM \cdot Ma : MP'^2 :: CA^2 : CB^2.$$

$$\text{Hence } MP' = MP.$$

COR. 2. If a circle be described on Aa as a diameter, (fig. Prop. VII.) and if the ordinate MP meet this circle in p ;

$$Mp^2 = AM \cdot Ma, \text{ by Eucl. III. 35. Hence}$$

$$Mp^2 : MP^2 :: AC^2 : BC^2; \text{ whence}$$

$$Mp : MP :: AC : BC.$$

PROP. V.

In the ellipse, if there be drawn through the focus a double ordinate Ll ; Ll is a third proportional to the axes major and minor. (See fig. p. 7.)

$$\text{By Prop. IV. } CA^2 : CB^2 :: AS \cdot Sa : SL^2.$$

$$\text{But } AS \cdot Sa = CA^2 - CS^2 = CB^2, \text{ by Prop. I. Cor. 3;}$$

$$\therefore CA^2 : CB^2 :: CB^2 : SL^2;$$

$$\text{whence } CA : CB :: CB : SL;$$

and taking the double of each term,

$$Aa : Bb :: Bb : Ll.$$

DEF. 5. The line Ll is called the *Latus Rectum* of the ellipse.

PROP. VI.

In the ellipse, if TPZ be a tangent at the point P , the angles SPT , HPZ are equal.

Let Q be a point near to P .

Take in SQ , $Sp = SP$, and take in HP , $Hq = HQ$, and join PQ .

Since by Def. $SP + PH = SQ$
+ QH , taking away QH ,

$$SP + PH - QH = SQ,$$

that is, $SP + Pq = SQ$;

whence $Pq = SQ - SP = Qp$.

Also, since $Sp = SP$, the angles SPp , SpP are equal; and ultimately, when Q comes up to P , and the angle PSQ vanishes, both these angles are right angles. Hence PpQ is ultimately a right angle. In like manner, QqP is ultimately a right angle.

Hence ultimately, by Int. Art. (12), the angles PQp , QPq are equal, that is, PQS , QPH are ultimately equal. Produce QP to any point X ; then, ultimately, when Q comes up to P , the secant QPX becomes a tangent ZPT by Int. (10); the angle PQS , or XQS , becomes TPS ; and the angle QPH becomes ZPH . Hence ZPT being the tangent, the angles TPS , ZPH are equal.

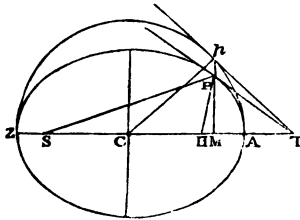
COR. If SP be produced to W , PW and PH make equal angles with PT the tangent. And in like manner if HP be produced.

PROP. VII.

In the ellipse, if PT , a tangent at P , meet the major axis produced in T , we have

$$CM : CA :: CA : CT.$$

For, by Prop. vi. Cor. since PT makes equal angles with SP produced and PH ,



we have, by Eucl. vi. A.

$HP : SP :: HT : ST$. Hence, by Introd. (4)

$HP + SP : HP - SP :: HT + ST : HT - ST$,

or $2CA : HP - SP :: 2CT : SH$. But by Introd. (7) and (8)

$HP - SP : HM - SM :: SH : HP + SP$ or $2CA$.

Compounding, $2CA : HM - SM$ or $2CM :: 2CT : 2CA$;

whence $CA : CM :: CT : CA$;

and $CM : CA :: CA : CT$.

COR. 1. If a circle be described on the major axis as a diameter, and if a tangent to this circle be drawn at the point where the ordinate meets it; the tangent to the ellipse and to the circle meet the major axis in the same point.

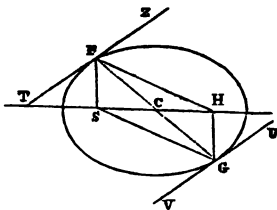
For in the circle, pT being the tangent, we have CpT a right angle, whence by Eucl. vi. 8, $CM : CA :: CA : CT$.

DEF. 3. Any line drawn through the center and terminated by the curve is a *Diameter*. The *Vertex* of a diameter is the point where it meets the curve.

PROP. VIII.

Every diameter is bisected by the center of the ellipse.

Let P be any point in the curve, C the center; join PC and produce it to G so that $CG = CP$; join SG , HG . Then by equal triangles PCS , GCH , $SP = HG$; in like manner $HP = SG$. Therefore $SG + GH = SP + PH$, whence by Def. G is a point in the curve; and therefore PG is a diameter; and PG is bisected in C .



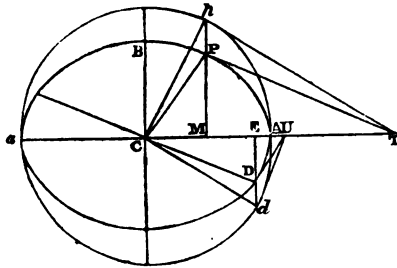
COR. 1. The angles SPC , HGC are equal, wherefore HG is parallel to SP . In like manner SG is parallel to HP ; and the figure $SPHG$ is a parallelogram.

COR. 2. The tangents at the extremities of a diameter are parallel. Let ZPT , VGU be the tangents at the extremities of the diameter PG . Then, since $HPZ = SPT$, $2SPT + SPH = 2$ right angles. In like manner $2HGU + SGH = 2$ right angles. Take away $SPH = SGH$. Hence $2SPT = 2HGU$, and $SPT = HGU$, or adding equals SPC , HGC , $CPT = CGU$, whence PT is parallel to GU .

DEF. 4. A second diameter is *conjugate* to a first, when it is parallel to the tangent drawn at the extremity of the first.

PROP. IX.

In the ellipse, if CD be conjugate to CP , CP is conjugate to CD .



Let a circle be described on the major axis Aa as a diameter; and let the ordinates MP , ED , at the points P and D , meet this circle in the points p and d . Let the tangents to the ellipse at P and D , meet the axis major in T and U ; then Tp and Ud will also be tangents to the circle at p and d , by Prop. VII. Cor. 1.

Since CD is conjugate to CP , by Def. 4, CD is parallel to PT ; whence by similar triangles,

$$MP : ED :: MT : CE.$$

Now we have $Mp^2 = Cp^2 - CM^2 = CA^2 - CM^2 = AM \cdot Ma$.

And $Ed^2 = AE \cdot Ea$, in like manner.

And by similar triangles, $CM : Mp :: Mp : MT$;

whence $CM \cdot MT = Mp^2$.

Hence $CM \cdot MT = AM \cdot Ma$.

But $AM \cdot Ma : MP^2 :: (CA^2 : CB^2 ::) AE \cdot Ea : ED^2$;
(Prop. iv.)

hence $CM \cdot MT : MP^2 :: Ed^2 : ED^2$.

and $CM \cdot MT : Ed^2 :: MP^2 : ED^2$,

that is $:: MT^2 : CE^2$, by sim. tri.

Hence, alt. $CM \cdot MT : MT^2 :: Ed^2 : CE^2$;

or $CM : MT :: Ed^2 : CE^2$;

or $CM^2 : CM \cdot MT :: Ed^2 : CE^2$;

whence, as before, $CM^2 : Mp^2 :: Ed^2 : CE^2$;

whence $CM : Mp :: Ed : CE$.

Hence the triangles CMp , dEC are similar; and the angle dCE is the complement of MCp ; whence pCd is a right angle.

Hence dU , perpendicular to Cd , is parallel to Cp ; and hence, by similar triangles, UEd , CMp , $UE : Ed :: CM : Mp$;

but we have also by Prop. iv. Cor. 2,

$Ed : ED :: (AC : BC ::) Mp : MP$;

hence compounding, $UE : ED :: CM : MP$.

Hence the triangles UED , CMp are similar, and hence CD is parallel to DU ; that is, CP is conjugate to CD .

COR. 1. $CM \cdot MT = AM \cdot Ma$.

COR. 2. CP , CD being conjugate, pCd is a right angle.

COR. 3. The triangles CMp , dEC are similar, and side $Cp = Cd$; hence they are equal, and $CM = Ed$. In like manner $CE = Mp$.

COR. 4. Also $DE : CM :: CB : CA$, because $CM = Ed$.

PROP. X.

In the ellipse, CP and CD being conjugate,
 $CP^2 + CD^2 = CA^2 + CB^2$.

$$\begin{aligned}\text{For (last fig.) } CP^2 + CD^2 &= CM^2 + MP^2 + CE^2 + ED^2 \\ &= CM^2 + MP^2 + Mp^2 + ED^2 \text{ by Prop. ix. Cor. 3.} \\ &= Cp^2 + MP^2 + ED^2 = CA^2 + MP^2 + ED^2.\end{aligned}$$

$$\begin{aligned}\text{But } Mp^2 : AC^2 &:: MP^2 : BC^2, \\ \text{and } Ed^2 : AC^2 &:: ED^2 : BC^2.\end{aligned}$$

Whence by Eucl. v. 24, $Mp^2 + Ed^2 : AC^2 :: MP^2 + ED^2 : BC^2$.

$$\begin{aligned}\text{But } Mp^2 + Ed^2 &= Mp^2 + CM^2 \text{ by Prop. ix. Cor. 3;} \\ \therefore MP^2 + ED^2 &= BC^2.\end{aligned}$$

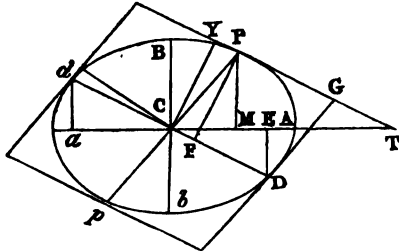
$$\text{Whence } CP^2 + CD^2 = CA^2 + CB^2.$$

PROP. XI.

The parallelograms circumscribed about an ellipse are all equal.

If Pp , Dd be two diameters conjugate to each other, and if at P , p , D , d , lines be drawn parallel to CD and CP , they will be tangents, by Prop. viii. Cor. 2, and form a circumscribed parallelogram.

And of this parallelogram $PCDG$ is one fourth part.



Let CY be perpendicular on PT ; then

$$\text{by Prop. vii, } CM : CA :: CA : CT;$$

$$\text{by sim. tri. } CD : DE :: CT : CY;$$

$$\text{by Prop. viii. Cor. 4, } DE : CM :: CB : CA.$$

$$\text{Hence compounding by Int. (5) } CD : CA :: CB : CY;$$

$$\text{whence } CD \cdot CY = CB \cdot CA.$$

And hence the parallelogram $CDGP$, which is equal to the rectangle, contained by CD , CY , is equal to the rectangle $CA \cdot CB$; and hence the whole circumscribed parallelogram is always = the rectangle $Aa \cdot Bb$.

COR. 1. Draw PF perpendicular to DC , then $PF = CY$; wherefore $CD \cdot PF = AC \cdot BC$.

COR. 2. Hence $AC : PF :: CD : BC$.

DEF. 5. The *Ordinate*, *Abscissa*, *Normal*, *Chords of Curvature*, and *Radius of Curvature*, are defined in the Ellipse in the same manner as in the Parabola. See the Definitions after Props. v. and viii. of the Parabola.

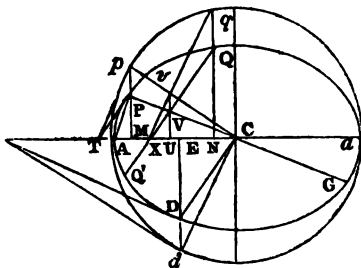
Thus if, in next figure, CP be a diameter, QV parallel to the tangent at P is an ordinate, and PV the abscissa.

PROP. XII.

In the ellipse, if QV be an ordinate to a diameter PCG , and CD conjugate to CP ,

$$PV \cdot VG : QV^2 :: CP^2 : CD^2.$$

Let Aa be the major axis of the ellipse; and on this, as a diameter, let a circle be described. Let the ordinates MP , NQ , meet the circle in p , q . Then by Prop. vi. Cor. 1. the tangents PT , pT meet the axis in the same point T . Let QV meet the axis in X , and Cp meet the ordinate UV in v .



Then $UV : Uv :: MP : Mp$ by sim. tri.

$$:: NQ : Nq \text{ by Prop. iv. Cor. 2.}$$

Therefore Xvq is a straight line.

Also by Prop. viii. Cor. 2. pCd is a right angle.

$$\text{Now } Cp^2 - Cv^2 = Cq^2 - Cv^2 = qv^2.$$

But $CP^2 : CV^2 :: Cp^2 : Cv^2$ by sim. tri.

$\therefore CP^2 : CP^2 - CV^2 :: Cp^2 : Cp^2 - Cv^2$, that is, qv^2 ;

or alt. $CP^2 : Cp^2 :: CP^2 - PV^2 : qv^2$.

But $Cd^2 : CD^2 :: (qX^2 : QX^2 ::) qv^2 : QV^2$ by sim. tri.

Compounding, recollecting that $Cp = Cd$;

$CP^2 : CD^2 :: CP^2 - PV^2 : QV^2$;

or, $CP^2 : CD^2 :: PV \cdot VG : QV^2$.

COR. If QV meet the ellipse again in Q' , we may prove in the same manner, that

$PV \cdot VG : Q'V^2 :: CP^2 : CD^2$; whence $Q'V = QV$.

PROP. XIII.

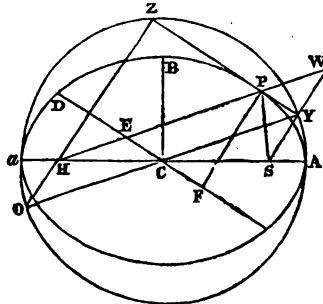
In the ellipse, the perpendiculars from the foci upon the tangent meet it in the circumference of a circle whose diameter is the axis major.

Let YPZ be a tangent at P , SY , and HZ perpendiculars upon it from the foci S , H .

Produce HP to W , making $PW = SP$. Join SW meeting the tangent in Y . Then because $PW = PS$, and $WPY = SPY$, (Prop. vi. Cor.) the triangles are equal in all respects, and SY is perpendicular to PY .

Join CY . Since $SW = 2SY$, and $SH = 2SC$, the triangles SCY , SHW are similar. Therefore $HW = 2CY$. But $HW = HP + PW = HP + PS = 2CA$.

Therefore $CY = CA$; and Y is in the circumference of the circle.



Similarly, by producing SP to v , making $Pv = PH$, and joining Hv meeting the tangent in Z , it may be shown that

HZ is perpendicular to the tangent, and *Z* a point in the circumference of the circle.

COR. 1. If *CD* be parallel to *PY*, meeting *OH* in *E*; *PECY* is a parallelogram, and *PE* = *CY*, and therefore *PE* = *AC*.

COR. 2. If *PS* produced meet *DC* in *E'*, *PE'* is equal to *PE* because the angles *PEE'*, *PE'E* are equal. Therefore *PE'* = *AC*.

PROP. XIV.

In the ellipse, the rectangle of perpendiculars from the foci upon the tangent is equal to the square of the semi-axis minor. (Last fig.)

Produce *ZH* to meet the circle (described on *Aa*) in *O*; join *CO*. Then, since *OZY* is a right angle, it is in a semicircle: therefore *YCO* is a diameter; and being a straight line, the triangles *YCS*, *OCH* are equal in all respects. Hence *HO* = *SY*: wherefore *SY.HZ* = *HO.HZ* = *AH.Ha* = *BC*², by Prop. 1. Cor. 6.

PROP. XV.

In the ellipse *SP.HP* = *CD*². (See fig. to Prop. XII.)

$$SP : SY :: PE : PF, \text{ by sim. tri.}$$

$$HP : HZ :: PE : PF, \text{ by sim. tri.}$$

$$\therefore SP.HP : SY.HZ :: PE^2 : PF^2$$

$$:: AC^2 : PF^2; \text{ by Prop. XIII. Cor. 1.}$$

$$\therefore SP.HP : SY.HZ :: CD^2 : BC^2, \text{ by Prop. XI. Cor. 2.}$$

$$\text{But } SY.HZ = BC^2, \text{ by Prop. XIV.}$$

$$\therefore SY.HP = CD^2.$$

PROP. XVI.

In the ellipse, if *PK* be the normal (see fig. Prop. XVII. supplying the line *PK*).

$$CM : MK :: CA^2 : CB^2.$$

Since PK is perpendicular to PT ,

By sim. tri. $TM : MP :: MP : MK$;

$$\therefore TM \cdot MK = MP^2.$$

But $TM \cdot MC = AM \cdot Ma$; by Prop. ix. Cor. 1.

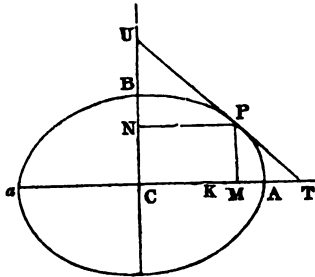
$$\therefore TM \cdot MC : TM \cdot MK :: AM \cdot Ma : MP^2$$

$$:: AC^2 : BC^2, \text{ by Prop. iv.}$$

$$\text{or } MC : MK :: AC^2 : BC^2.$$

PROP. XVII.

In the ellipse, if PU be a tangent at P meeting the minor axis produced at U , and if PN be perpendicular on the minor axis; $CN : CB :: CB : CU$.



For by Prop. vii. $CM : CA :: CA : CT$.

Hence $CA^2 = CM \cdot CT$: whence

$$CA^2 - CM^2 = CM \cdot CT - CM^2 = CM \cdot MT.$$

But $MT : CT :: MP : CU$ by similar triangles: whence

$$CM \cdot MT : CM \cdot CT :: MP^2 : CU \cdot MP; \text{ that is,}$$

$$CA^2 - CM^2 : CA^2 :: MP^2 : CU \cdot MP; \text{ or, alternando,}$$

$$AM \cdot Ma : MP^2 :: CA^2 : CU \cdot CN. \text{ But by Prop. iv.}$$

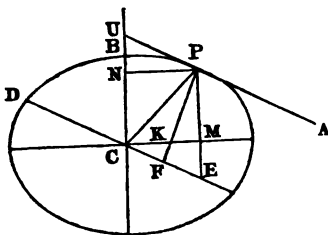
$$AM \cdot Ma : MP^2 :: CA^2 : CB^2; \text{ wherefore}$$

$$CU \cdot CN = CB^2; \text{ whence } CN : CB :: CB : CU.$$

PROP. XVIII.

In the ellipse, if PK be a normal, meeting the major axis in K and the conjugate in F , $PK \cdot PF = BC^2$.

Draw PM perpendicular to the major axis in M , meeting the conjugate in E . Then since the conjugate CE is parallel to the tangent PU , it is evident that $PE = CU$. Also $PM = CN$. And the triangles PMK , PFE are similar, having a common angle at P , and right angles.



Therefore

$$PK : PM :: PE : PF;$$

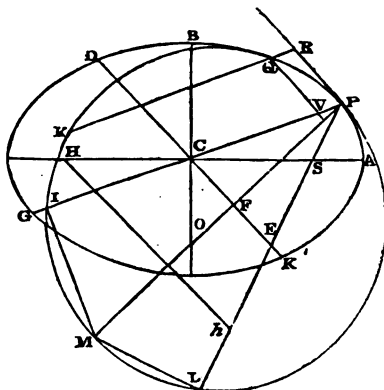
whence $PK \cdot PF = PM \cdot PE = CN \cdot CU = CB^2$,
by Prop. xvii.

PROP. XIX.

In the ellipse, if PI be the chord of curvature at the point P passing through the center,

$$CP : CD :: 2CD : PI.$$

Let PQK be a circle which has the tangent PR in com-



mon with the ellipse at P , and which cuts the ellipse at Q , near to P . Draw RQK , parallel to PC , and let CP cut the circle in I . Draw QV parallel to RP . Then by Eucl. III. 36. $RK \cdot RQ = RP^2$, that is, $= QV^2$. But by Prop. XII.

$$PV \cdot VG : QV^2 :: CP^2 : CD^2; \text{ therefore}$$

$$PV \cdot VG : RK \cdot RQ :: CP^2 : CD^2; \text{ but } PV = RQ;$$

$$\therefore VG : RK :: CP^2 : CD^2.$$

And ultimately, when Q comes up to P , VG becomes PG , or $2CP$, (Prop. VIII.) and RK becomes PI , the chord of curvature.

Therefore $2CP : PI :: CP^2 : CD^2$, whence

$$2CP^2 : CP \cdot PI :: 2CP^2 : 2CD^2; \text{ therefore}$$

$$CP \cdot PI = 2CD^2 = CD \cdot 2CD; \text{ whence}$$

$$CP : CD :: 2CD : PI.$$

COR. 1. If PM be the diameter of the circle of curvature, let PM meet CD in F ; then since PM is perpendicular to the tangent at P , it is perpendicular to CD , and PFC is a right angle. Hence the triangles PCF , PMI are similar;

and $CP : PF :: PM : PI$, whence $CP \cdot PI = PF \cdot PM$.

Hence by the proof of the Prop. $PF \cdot PM = 2CD^2$; or, since $PM = 2PO$, PO being the radius of curvature,

$$2PF \cdot PO = 2CD^2, \text{ } PF \cdot PO = CD^2,$$

$$\text{whence } PF : CD :: CD : PO.$$

COR. 2. Let PSL be the chord of the circle of curvature through the focus S . Let PL meet CD in E , and join PH ; then $PE = AC$; Prop. XIII. Cor. 1.

Join ML . Then PFE , PLM are similar triangles;

$$\text{therefore } PM : PL :: PE : PF;$$

$$\text{hence } PL \cdot PE = PM \cdot PF = 2CD^2 \text{ by Cor. 1;}$$

$$\text{or } PL \cdot AC = 2CD^2.$$

Hence $AC : CD :: 2CD : PL$;

and PL is the chord of curvature through the focus.

THE HYPERBOLA.

DEF. 1. A Hyperbola is a curve line such that every point in the curve has the difference of its distances from two fixed points (the *Foci*) equal to a constant line.

If S, H be the two fixed points; P a point in the curve: then $HP - SP =$ a constant line, wherever be P .

PROP. I.

If SH be joined and bisected in C , and if A in HS be a point in the curve, $HP - SP = 2CA$.

For since A, P , are points in the curve,

$$\begin{aligned} HP - SP &= HA - SA = HC + CA - SA \\ &= SC - SA + CA = CA + CA = 2CA. \end{aligned}$$

COR. 1. If a , also in SH , be a point at which

$$Sa - Ha = HP - SP,$$

a will be a point in a curve pa , in which $Sp - Hp =$ the same constant line as $HP - SP$. In this case

$$Sp - Hp = 2Ca.$$

The curve pa is the *Opposite Hyperbola*.

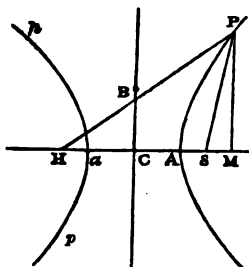
COR. 2. Hence $HP - SP = Aa$.

DEF. 2. C is the *Centre* of the hyperbola. Aa is called the *Axis Major*, or *Transverse Axis*; CA is called the *Semi-axis Major*. The ratio of CS to CA is called the *Excentricity*.

DEF. 3. Let CB be perpendicular to CA , and such that

$$CB^2 = CS^2 - CA^2.$$

Then by analogy with the ellipse, CB is called the *Semi-axis Minor* of the hyperbola, or the *Conjugate Semi-axis*.



Draw EK perpendicular to SE and PK parallel to SE ,
then $CE : CA :: CA : CS$;

$\therefore CE : CA :: CM : CI$; (by Prop. II.)

$\therefore CE : CM :: CA : CI$;

$\therefore CE : CM - CE :: CA^2 : CI - CA$;

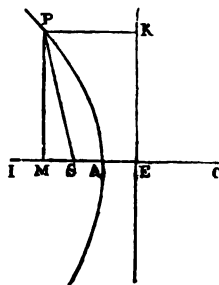
$\therefore CE : ME :: CA : AI$;

$\therefore CD : PK :: CA : SP$; (by Prop. II.)

$\therefore SP : PK :: (CA : CD) :: CS : CA$.

Hence $SP : PK$ is a constant ratio.

DEF. 4. EK is called the *Directrix* of the hyperbola.



PROP. IV.

In the hyperbola, MP being an ordinate to the axis major,

$AM.Ma : MP :: CA^2 : CB^2$. (Fig. Prop. II.)

Take $CA : CS :: CM : CI$; whence $CA.CI = CS.CM$.

Now by Prop. II. $SP = AI$, whence $SP^2 = AP^2$.

But $SP^2 = SM^2 + MP^2$; $\therefore AI^2 = SM^2 + MP^2$;

that is $(CI - CA)^2 = (CM - CS)^2 + MP^2$;

hence $CI^2 + CA^2 - 2CA.CI = CM^2 + CS^2 - 2CS.CM + MP^2$.

And, adding the equals, $2CA.CI$ and $2CS.CM$;

$CI^2 + CA^2 = CM^2 + CS^2 + MP^2$.

But $CS^2 = CA^2 + CB^2$ by Prop. I. Cor. 2;

$\therefore CI^2 + CA^2 = CM^2 + CA^2 + CB^2 + MP^2$,

$\therefore CI^2 = CM^2 + CB^2 + MP^2$,

$CI^2 - CM^2 = CB^2 + MP^2$.

But since $CA : CS :: CM : CI$, $CA^2 : CS^2 :: CM^2 : CI^2$;

whence $CS^2 : CA^2 :: CI^2 : CM^2$,

and $CS^2 - CA^2 : CA^2 :: CI^2 - CM^2 : CM^2$;

or, Prop. 1. Cor. 2, $CB^2 : CA^2 :: CB^2 + MP^2 : CM^2$;

hence $CB : CB^2 + MP^2 :: CA^2 : CM^2$;

whence $CB^2 : MP^2 :: CA^2 : CM^2 - CA^2$;

hence alt. $CB^2 : CA^2 :: MP^2 : AM \cdot Ma$;

whence $AM \cdot Ma : MP^2 :: CA^2 : CB^2$.

COR. 1. If MP' be the ordinate on the other side of the axis, $MP' = MP$.

COR. 2. If there be drawn through the focus a double ordinate Ll , Ll (which is called the *Latus Rectum*) is a third proportional to the axis major and minor.

The proof is the same as that of Prop. v. in the Ellipse.

PROP. V.

In the hyperbola, if AD , Ad be parallel and equal to CB , and if MP meet CD , Cd in R , r ;

$$RP \cdot Pr = CB^2.$$

By similar triangles,

$CA : CB$, that is $AD :: CM : MR$.

Hence $CA^2 : CB^2 :: CM^2 : MR^2$.

But $CM^2 - CA^2 : MP^2 :: CA^2 : CB^2$;
Prop. iv.

$$\therefore CM^2 - CA^2 : MP^2 :: CM^2 : MR^2;$$

$$\therefore \text{alt. } CM^2 - CA^2 : CM^2 : MP^2 : MR^2;$$

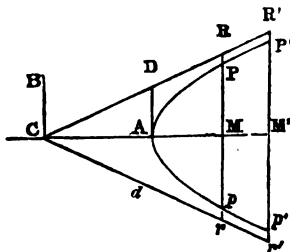
$$\therefore CM^2 - CA^2 : CA^2 :: MP^2 : MR^2 - MP^2;$$

$$\therefore CM^2 - CA^2 : MP^2 :: CA^2 : MR^2 - MP^2.$$

$$\text{But } CM^2 - CA^2 : MP^2 :: CA^2 : CB^2;$$

$$\therefore MR^2 - MP^2 = CB^2.$$

$$\text{But } MR^2 - MP^2 = RP \cdot Pr; \therefore RP \cdot Pr = CB^2.$$



COR. 1. If $M'P'$ be any other ordinate, meeting CD , Cd , in R' , r' ; $R'P' \cdot P'r' = RP \cdot Pr$.

COR. 2. As CM increases without limit, Pr increases without limit, and PR diminishes without limit.

Hence as P is further from A , the curve approaches nearer to the straight line CR , without limit, but never meets it.

DEF. 5. CR (and in like manner Cr) is called an *Asymptote* to the hyperbola.

COR. 3. Since $RP \cdot Pr = Rp \cdot pr$, it is evident that $PR = pr$: for the given line Rr being divided into segments making a given rectangle, the magnitude of the segments is given.

PROP. VI.

In the hyperbola, if any line TQt meet the asymptotes and the curve; and if CV , bisecting Tt , meet the curve in P , and PL , parallel to Tt , meet the asymptote in L ,

$$TQ \cdot Qt = PL^2.$$

For since V bisects Tt , it is easily shown, by the similar triangles CVT , CPL , and CVt , CPl , that P bisects Ll . And SQs , RPr , being perpendicular to the principal axis; we have, also by similar triangles;

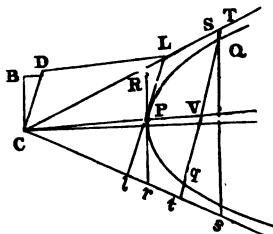
$$TQ : SQ :: PL : RP,$$

$$Qt : Qs :: Pl : Pr;$$

$$\therefore TQ \cdot Qt :: SQ \cdot Qs :: PL^2 : RP \cdot Pr.$$

But $SQ \cdot QS = RP \cdot Pr$, by (Prop. v. Cor. 1.);

$$\therefore TQ \cdot Qt = PL^2.$$



COR. 1. It appears in the same manner that if TQ meet the curve again in q , $Tq \cdot qt = PL^2$. Hence $TQ \cdot Qt = Tq \cdot qt$. Therefore $QT = qt$: for the given line Tt being divided into segments making a given rectangle, the magnitude of the segments is given.

COR. 2. Hence the line LPl meets the curve in one point P only; therefore PL is a tangent at P (Int. Art. 10.) Hence all lines such as TQt are parallel to the tangent at P .

DEF. 6. In the hyperbola, as in the ellipse, a *Diameter* is a line passing through the center and terminated by the curve. The *Vertex* is the point where the diameter meets the curve. An *Ordinate* to any diameter is a line parallel to the tangent at the Vertex, and meeting the diameter: the *Abcissa* is the line cut off from the diameter by the ordinate.

COR. 3. It appears by the proof of Prop. VI., that if an ordinate QV meet the curve again in q , $qV = QV$.

COR. 4. Also that if a tangent P meet the asymptotes in L, l , $Pl = PL$.

PROP. VII.

In the hyperbola, a tangent at any point bisects the angle made by drawing two lines from the point to the foci.

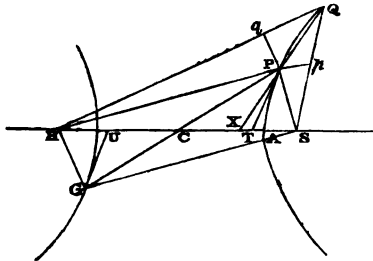
Let P be the point in the curve; PS, PH the lines drawn to the foci; Q a point near to P . Take in SQ , $Sp = SP$, and in HQ , $Hq = HP$. Then $HQ - SQ = HP - SP$,

whence $HQ = HP + SQ - SP$
 $= HP + Qp$, and hence

$$HQ - HP = Qp,$$

that is, $Qq = Qp$.

Also since $Sp = SP$, the angles SPp, SpP are equal; and ultimately when Q comes up to P , and the angle PSQ vanishes, both these angles are right angles. Hence PpQ is ultimately a right angle. In like manner PqQ is ultimately a right angle. Hence ultimately, by Int. (12) the angles PQp, PQq are equal, that is, PQS, PQH , or XPS, XPH are ultimately equal. But ultimately



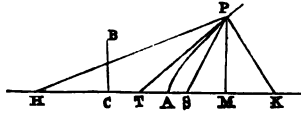
the secant XQP becomes a tangent PT . Hence ultimately the angles TPS , TPH are equal.

PROP. VIII.

In the hyperbola, if PT , a tangent at P , meet the major axis in T , and PM be the ordinate; we have

$$CM : CA :: CA : CT.$$

For since PT bisects the angle HPS , we have by Eucl. vi. 3. $HP : SP :: HT : ST$.



Hence, Int. (3)

$$HP + SP : HP - SP :: HT + ST : HT - ST,$$

$$\text{or, (Prop. 1.) } HP + SP : 2CA :: SH : 2CT.$$

But by Introd. (7) or (8), PM being perpendicular on HS ,

$$HM + SM : HP + SP :: HP - SP, \text{ or } 2CA : SH.$$

Compounding, $HM + SM$, or $2CM : 2CA :: 2CA : 2CT$;

$$\text{whence } CM : CA :: CA : CT.$$

$$\text{COR. Hence } CM \cdot CT = CA^2.$$

PROP. IX.

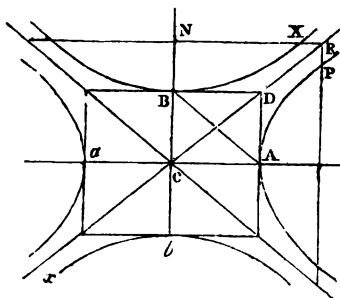
In the alternate angles between the asymptotes of two opposite hyperbolas may be described two other hyperbolas having the same asymptotes.

The semi-axis minor CB being perpendicular to CA , as in Def. 3, and any line NX , parallel to CA , being drawn, if X be always taken so that

$$CN^2 - CB^2 : NX^2 :: CB^2 : CA^2;$$

the curve BX will have the same property which AP is shown to have in Prop. v., and hence will be an hyperbola. And in like manner, if $Cb = CB$ on the other side of CA , bx drawn in the same way as bx , will be the opposite hyperbola to BX .

And if AD be equal and parallel to CB , CD will be an asymptote to the curves, by Prop. v. Cor. 2.

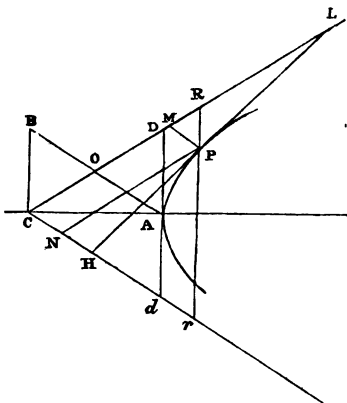


DEF. 6. BX , bx are conjugate hyperbolas to AP and ap .

PROP. X.

In the hyperbola, if from any point P , PM , PN be drawn parallel to the asymptotes and meeting them in M and N , $PM \cdot PN$ is the same for every point P .

Let the asymptotes CD , Cd be drawn as in Prop. v., and let PM , PN be drawn parallel to Cd , CD , and meeting CD , Cd in M , N . Join AB , meeting CD in O ; AB will be parallel to Cd , or PM ; and $ABCD$ will be a rectangle. Through P , let RP meet the asymptotes in R , r .



Then by similar triangles,

$$PM : PR :: AO : AD;$$

$$PN : Pr :: OD : AD.$$

Compounding,

$$PM \cdot PN : PR \cdot Pr :: AO \cdot OD : AD^2.$$

$$\text{But by Prop. v. } RP \cdot Pr = CB^2 = AD^2.$$

$$\text{Hence } PM \cdot PN = AO \cdot OD = AO^2:$$

for $AO = OD$, since $ADBC$ is a rectangle.

And hence $PM \cdot PN$ is the same for every point P .

COR. 1. We have also $CM \cdot MP = AO^2$.

COR. 2. The parallelogram $CMPN$ is double of the triangle COA . For if CP be joined, the triangles CMP , COA are equal, since they have the angles CMP , COA equal and the sides reciprocally proportional. (Eucl. vi. 15.)

COR. 3. The parallelograms $CMPN$ are equal, wherever is P . (Eucl. vi. 14.)

COR. 4. If LPH , a tangent at P , meet the asymptotes in L and H , the triangles LCH are equal wherever is P . For by Prop. vi. Cor. 5, LH is bisected in P . Hence $LM = MC$, and the triangle LCH is double of the parallelogram $CMPN$.

COR. 5. $CM : CO :: AO : MP$.

PROP. XI.

Every diameter is bisected by the center of the hyperbola.

Let P be any point in the curve (fig. to Prop. vii.), C the center. Join PC , and produce it to G , so that $CG = CP$: join SG , HG . Then by equal triangles PCS , GCH , $SP = HG$; in like manner, $HP = SG$. Therefore $SG - HG = HP - SP$: whence by Prop. i. Cor. 1, G is a point in the opposite hyperbola. And hence PG is a diameter. And PG is bisected in C .

COR. 1. The figure $SPHG$ is a parallelogram.

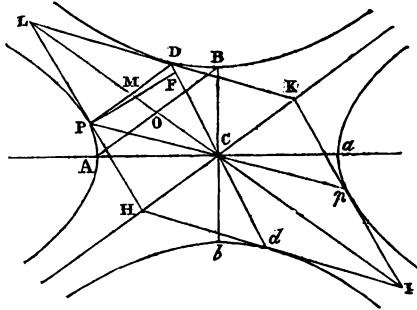
COR. 2. The tangents at the extremities of a diameter are parallel. Let PT , GU be the tangents at the extremities P , G of the diameter PG . Then because $CPS = TPS + CPT$, and $CPH = TPH - CPT$, we have $CPS - CPH = 2CPT$, because TPS and TPH are equal. In like manner $CGH - CGS = 2CGU$, because UGH and HGS are equal. But because $CPS = CGH$, and $CPH = CGS$, we have $CPS - CPH = CGH - CGS$. Therefore $2CPT = 2CGU$, and $CPT = CGU$; whence PT is parallel to GU .

DEF. 7. In next figure, if CD be parallel to the tangent at P , the Diameter CD is *conjugate* to the Diameter CP .

PROP. XII.

In the hyperbola, the tangents at the extremities of conjugate diameters form a parallelogram of which the asymptotes are diagonals.

Draw PMD parallel to the asymptote HCK , meeting



the hyperbola and the conjugate hyperbola in P, D . Then, Prop. x. Cor. 1, $CM \cdot MP = AO^2$. For the same reason $CM \cdot MD = BO^2$. But $AO = BO$. Therefore $MD = MP$.

Take $ML = CM$, and draw LPH , LDK meeting the asymptote HCK . Then by similar triangles, $LP = PH$. Hence by Prop. vi. Cor. 4, PL is a tangent at P . For the like reason DL is a tangent at D .

But since $HP = PL$, and $KD = DL$, CP, CD are parallel to LK, LH , and $CPLD$ is a parallelogram. And in like manner Pd, Dp, dp are parallelograms, and $HIKL$ is a parallelogram. And $Cp = CP, Cd = CD$.

And hence, by Def. 7, CP, CD are evidently conjugate each to the other. Also by Prop. xi., since C bisects Pp, Dd , p, d , are points in the opposite and conjugate hyperbolas respectively; and because $Ip = pK$, IK is a tangent, and because $Hd = dI$, HI is a tangent. Hence the parallelogram $HIKL$ is formed by tangents at the extremities of conjugate diameters.

COR. 1. If CD is conjugate to CP , CP is conjugate to CD .

COR. 2. PD parallel to the asymptote, is bisected in M .

COR. 3. In Prop. vi. PL is equal to CD , the semi-conjugate to CP ; and that Proposition becomes $TQ \cdot Qt = CD^2$.

PROP. XIII.

In the hyperbola, the parallelograms formed by the tangents at the extremities of conjugate diameters are all equal.

For by Prop. x. Cor. 2, the triangle HCL is equal wherever is P . And the parallelogram $HIKL$ in Proposition XII. is four times the triangle HCL . Therefore the parallelogram $HIKL$ is always the same in area.

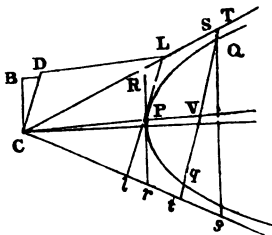
COR. 1. If PF be perpendicular on CD , $CD \cdot PF = AC \cdot BC$; for $CD \cdot PF$ is the parallelogram $CPDL$, and $AC \cdot BC$ is the corresponding parallelogram when the conjugate diameters are CA , CB .

COR. 2. Hence $AC : PF :: CD : BC$.

PROP. XIV.

In the hyperbola, if CP be any diameter and QV an ordinate to it,

$$PV \cdot VG^* : QV^2 :: CP^2 : CD^2.$$



By Prop. vi. $TQ \cdot Qt = PL^2$,
that is, since V bisects Tt , $VT^2 - VQ^2 = PL^2$;

$$\therefore VT^2 - PL^2 = VQ^2.$$

But by sim. tri. $CV : CP :: VT : PL$;

$$\therefore CV^2 : CP^2 :: VT^2 : PL^2;$$

$$\therefore CV^2 - CP^2 : CP^2 :: VT^2 - PL^2 : PL^2,$$

* In the figure, PC should be produced to a point G , so that $CG = CP$.

$$\therefore SP \cdot HP : SY \cdot HZ :: PE^2 : PF^2$$

$$:: AC^2 : PF^2 \text{ by Prop. xvi. Cor.}$$

$$DC^2 : BC \text{ by Prop. xiii. Cor. 2.}$$

$$\text{But } SY \cdot HZ = BC^2, \text{ by Prop. xvii.}$$

$$\therefore SP \cdot HP = CD^2.$$

PROP. XIX.

In the hyperbola, if PK be the normal,

$$CM : MK :: CA^2 : CB^2. \text{ (See fig. Prop. viii.)}$$

We have $CM^2 = CM \cdot CT + CM \cdot MT$ by Eucl. II.

$$= CA^2 + CM \cdot MT, \text{ by Prop. viii. Cor. 1;}$$

$$\therefore CM^2 - CA^2 = CM \cdot MT;$$

$$\text{that is, } AM \cdot Ma = CM \cdot MT. \text{ Eucl. II. 4.}$$

$$\text{Now by sim. tri. } MT : MP :: MP : MK;$$

$$\therefore MK \cdot MT = MP^2.$$

$$\text{Hence } CM \cdot MT : MK \cdot MT :: AM \cdot Ma : MP^2;$$

$$\text{or } CM : MK :: CA^2 : CB^2.$$

PROP. XX.

In the hyperbola, if PI be the chord of curvature at the point P , passing through the center,

$$CP : CD :: 2CD : PI.$$

Let PQK be the circle which has the tangent PR converse with the ellipse at P , and which cuts the curve again in Q . Draw RQK , parallel to CP , and let CP cut the circle in I . If QV

$$2PF \cdot PO = 2CD^2, \quad PF \cdot PO = CD^2; \text{ whence}$$

$$PF : CD :: CD : PO.$$

COR. 2. Let PSL be the chord of curvature through the focus S . Let PL meet CD in E , and join PH ; then $PE = AC$, by Prop. xvi. Cor.

Join ML , then PFL , PLM are similar triangles.

Therefore $PM : PL :: PE : PF$;

hence $PL \cdot PE = PM \cdot PF = 2CD^2$, by Cor. 1,

or $PL \cdot AC = 2CD^2$.

Hence $AC : CD :: 2CD : PL$;

And PL is the chord of curvature through the focus.

THE END.

